

## Problem Set # 11

**Due Wednesday december 4th in recitation**

### Exercise 1

Expand  $(x + i^k y, x + i^k y) = \|x + i^k y\|^2$  to verify the polarization identities.

### Exercise 2

Let  $T : V \rightarrow W$  be a linear operator between finite-dimensional inner product spaces and let  $\mathcal{X} = \{e_i\}$ ,  $\mathcal{Y} = \{f_i\}$  be ON bases. Prove that  $[T]_{\mathcal{Y}\mathcal{X}}$  has entries

$$T_{ij} = (T_{e_j}, f_i)_W = (f_i, T(e_j))_W, \text{ for } 1 \leq i \leq \dim_K(W), 1 \leq j \leq \dim_K(V)$$

### Exercise 4

- Let  $V = V_1 \oplus \dots \oplus V_r$  be a direct sum of orthogonal subspaces of an inner product space (so  $V_i \perp V_j$ , for  $i \neq j$ ). Prove that
  - $V_i \perp (\sum_{j \neq i} V_j)$  for each  $i$ ;
  - $\|v\|^2 = \sum_i \|v_i\|^2$  if  $v = v_1 + \dots + v_r$  with  $v_i \in V_i$ .
- Show that  $M^\perp$  is again a subspace of  $V$ , and that

$$M_1 \subseteq M_2 \Rightarrow M_2^\perp \subseteq M_1^\perp$$

### Exercise 6 (Gram-Schmidt construction)

- Let  $v_1, \dots, v_n$  be independent vectors in an inner product space and let  $M_k = K - \text{Span}\{v_1, \dots, v_k\}$ , for  $k = 1, 2, \dots, n$ . Via an inductive algorithm, construct orthonormal vectors  $e_1, \dots, e_n$  such that
  - $e_k \in K - \text{Span}\{v_1, \dots, v_k\}$ ;
  - $K - \text{Span}\{e_1, \dots, e_n\} = K - \text{Span}\{v_1, \dots, v_n\}$ .

The result is an orthonormal basis  $\{e_1, \dots, e_n\}$  for  $M = K - \text{Span}\{v_1, \dots, v_n\}$ .

- Consider  $\mathcal{C}[-1, 1]$  together with the inner product  $(f, h) = \int_{-1}^1 f(t) \overline{h(t)} dt$ . Take  $v_1 = 1$ ,  $v_2 = x$ ,  $v_3 = x^2$  regarded as functions  $f : [-1, 1] \rightarrow \mathbb{C}$ . Find the orthonormal set  $\{e_1, e_2, e_3\}$  produced by Gram-Schmidt algorithm described in the previous question.

**Exercise 5 (Fourier series))**

1. Consider  $\mathcal{C}[0, 1]$  together with respect to the inner product  $(f, h) = \int_0^1 f(t)\overline{h(t)}dt$ . Define  $e_n(t) = e^{2\pi i n t}$  ( $n \in \mathbb{Z}$ ) Prove that  $\{e_n\}$  is an orthonormal set.

**Exercise 6**

If  $\{e_i\}$  is orthonormal, and  $p_M(v) = \sum_{i=1}^N (v, e_i)e_i$  the projection of  $v$  onto  $M = K - \text{Span}\{e_1, \dots, e_N\}$  along  $M^\perp$ . Prove that the following geometric property holds

$$\begin{aligned} p_M(v) &= \text{the point in } M \text{ closest to } v \text{ in the sense that} \\ \|p_M(v) - v\| &= \min\{\|u - v\| \mid u \in M\} \end{aligned}$$

for any  $v \in V$ . In particular the minimum is achieved at a unique point  $p_M(v) \in M$ .